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Lie algebra deformation of $Diff_A T^2$ and Bloch electrons in a constant uniform magnetic field

T Dereli and A Verçin

Department of Physics, Faculty of Sciences, Ankara University, 06100 Ankara, Turkey

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Abstract. Magnetic translation operators that generate symmetries of a Bloch electron in a constant uniform magnetic field are shown to span the sine bracket algebra which is the unique Lie algebra deformation of the area-preserving diffeomorphisms of a two-torus. The deformation parameter is identified with the number of magnetic flux quanta through a unit cell of the substrate lattice. The symmetries of a spin-1/2 Bloch electron are shown to realize the supersymmetric extension of the sine bracket algebra.

1. Introduction

Infinite-dimensional algebras that generate area-preserving diffeomorphisms of a two-dimensional surface were first studied by Arnold [1] with applications in the hydrodynamics of incompressible fluids. The recent interest in these algebras is mostly due to their relevance as residual symmetries of relativistic membranes [2–6]. In general, symmetry generators of a physical system such as a membrane with infinitely many degrees of freedom may be thought of as the limiting case of finite-dimensional models. A way of determining an infinite extension of a finite-dimensional Lie algebra consists of requiring the structure constants in some basis of the infinite-dimensional algebra to converge to those of the limiting finite Lie algebra. This process is called a deformation of the Lie algebra [7]. In this paper we will concentrate on the following Lie algebras:

$$[T_m, T_n] = \frac{N}{2\pi} \sin\left(\frac{2\pi}{N} \boldsymbol{m} \times \boldsymbol{n}\right) T_{m+n} \tag{1}$$

$$[\mathrm{L}_m, L_n] = \boldsymbol{m} \times \boldsymbol{n} L_{m+n} \tag{2}$$

where $\boldsymbol{m}$ and $\boldsymbol{n}$ are elements of a two-dimensional integral lattice, and N is an odd integer. Thus we may identify the generators $T_{m+(Np,Nq)}$ with T_m for any $p, q \in Z$. In the limit $N \to \infty$ the sine algebra (1) converges to the algebra (2). The above algebras found applications in (i) hydrodynamics, (ii) the trigonometric solutions of the Yang–Baxter equations, (iii) relativistic membranes, and (iv) atomic spectroscopy. We show here that the problem of Bloch electrons in a uniform magnetic field [8] admits the supersymmetric generalization of the sine algebra (1) as the symmetry algebra.

The paper is organized as follows. In section 2, a brief review of the area-preserving diffeomorphisms of a 2-torus $Diff_A T^2$ and $SU(N)$ algebras is given. In section 3 the motion of Bloch electrons in a constant uniform magnetic field is studied. It is shown that the magnetic translation generators span sine algebra. We give a physical realization of the supersymmetric sine algebra in section 4. The final section is devoted to a discussion of results.

2. $Diff_A T^2$ and $SU(N)$ algebras

The vector fields that generate the diffeomorphisms of a 2-torus are

$$L_f = \frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x} \tag{3}$$

$$\tilde{\eta} = c_1 \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y} \tag{4}$$

in terms of canonical coordinates $(x, y) \in [0, 2\pi) \times [0, 2\pi)$, where c_1 and c_2 are arbitrary constants. These generators satisfy the commutation relations

$$[L_f, L_g] = L_{\{g,f\}} \tag{5}$$

$$[\tilde{\eta}, L_f] = L_{\tilde{\eta} f} \tag{6}$$

Let us consider an orthogonal basis for N defined by the eigenfunctions

$$Y_m = \mathrm{e}^{\mathrm{i}(m_1 x + m_2 y)} \tag{7}$$

where $\boldsymbol{m} = (m_1, m_2) \in \boldsymbol{Z} \times \boldsymbol{Z}$. The Lie commutator algebra can be completely characterized by the structure constants calculated in the basis (7). The Poisson bracket of Y_m and Y_n is given by

$$\{Y_m, Y_n\} = -\boldsymbol{m} \times \boldsymbol{n} Y_{m+n}. \tag{8}$$

Then, writing for the algebra generators $L_m = L_{Y_m}$ and taking into consideration also the generators $P_1 = \partial/\partial x$ and $P_2 = \partial/\partial y$ corresponding to the harmonic form η, we determine the complete algebra of area-preserving diffeomorphisms on a 2-torus:

$$[L_m, L_n] = \boldsymbol{m} \times \boldsymbol{n} L_{m+n} \tag{9}$$

$$[P_i, P_j] = 0 \qquad i, j = 1, 2 \tag{10}$$

$$[P_j, L_m] = i m_j L_m. \tag{11}$$

In the following we will be interested only in the invariant subalgebra generated by L_m's, denoted by $Diff_A T^2$. The infinite sine algebra (1) arises as unique Lie algebra deformation of $Diff_A T^2$ in some suitable basis. We refer to the work of Fairlie *et al* [4] for the actual demonstration of the fact.

It is well known that the structure constants of $Diff_A T^2$ can be approximated by $SU(N)$ structure constants. To demonstrate this the following $N \times N$ matrices are used [9]:

$$h = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \dots \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \omega & 0 & \dots \\ 0 & 0 & \omega^2 & \dots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \omega^{N-1} \end{pmatrix}. \tag{12}$$

These matrices satisfy

$$h^N = 1 \qquad g^N = 1 \qquad hg = \omega gh \tag{13}$$

where ω is an Nth root of unity with period no smaller than N. Let $\omega = e^{4\pi i/N}$, where N is odd. If to each vector $\boldsymbol{m} = (m_1, m_2)$ we assign the $N \times N$ matrix

$$K_{\boldsymbol{m}} = N\omega^{\,m_1 m_2/2} g^{m_1} h^{m_2} \tag{14}$$

and using (12) we can show that

$$K^{\dagger}_{\boldsymbol{m}} = K_{-\boldsymbol{m}} \qquad \operatorname{tr} K_{\boldsymbol{m}} = 0 \tag{15}$$

except for $m_1 = m_2 = 0 \bmod N$, and

$$K_{\boldsymbol{m}} = K_{\boldsymbol{n}} \Leftrightarrow m_i = n_i \bmod N. \tag{16}$$

Thus the vectors can be restricted to the sublattice defined by $m_1, m_2 = 0, 1, 2, \ldots, N-1$. If the origin $\boldsymbol{m} = (0, 0)$ is excluded, then there are precisely $N^2 - 1$ traceless, unitary, independent matrices $K_{\boldsymbol{m}}$ which can be used as the generators of $SU(N)$ algebra. On the other hand if the origin is included, then we have a complete set of $N \times N$ matrices which close under multiplication

$$K_{\boldsymbol{m}} K_{\boldsymbol{n}} = N\omega^{-\boldsymbol{m}\times\boldsymbol{n}/2} K_{\boldsymbol{m}+\boldsymbol{n}} \quad \operatorname{tr}(K_{\boldsymbol{m}} K_{\boldsymbol{n}}) = N^3 \delta_{\boldsymbol{m}+\boldsymbol{n}} \tag{17}$$

where $\delta_{\boldsymbol{m}+\boldsymbol{n}}$ equals one if and only if $m_i = n_i \bmod N$, and zero otherwise. Then one easily derives the algebra

$$[K_{\boldsymbol{m}}, K_{\boldsymbol{n}}] = 2iN \sin\left(\frac{2\pi}{N}\, \boldsymbol{m} \times \boldsymbol{n}\right) K_{\boldsymbol{m}+\boldsymbol{n}} \tag{18}$$

which in the limit $N \to \infty$ converges (up to a scale factor) to the algebra of area-preserving diffeomorphisms of the 2-torus:

$$\lim_{N\to\infty} [K_{\boldsymbol{m}}, K_{\boldsymbol{n}}] = 4\pi i [L_{\boldsymbol{m}}, L_{\boldsymbol{n}}]. \tag{19}$$

Therefore, the structure constants of the algebra $Diff_A T^2$ can be approximated by those of $SU(N)$ and moreover the large N corrections can be determined explicitly from the above expression.

3. Bloch electrons in a constant uniform magnetic field

In this section we show explicitly that the symmetry operators of Bloch electrons in a constant uniform magnetic field generate the sine algebra (1) and in the limit $\alpha \to 0$ approaches the algebra $Diff_A T^2$, where α is the number of magnetic flux quanta passing through a unit cell of the substrate lattice. In order to clarify these statements, we start with a non-relativistic electron of charge e, mass μ moving on a Bravais lattice in the xy-plane under the influence of a constant uniform magnetic field $\boldsymbol{B} = B\hat{z}$. The Hamiltonian is

$$H = \frac{\delta_{ij}}{2\mu}\, \pi^i \pi^j + V(x, y) \tag{20}$$

where the substrate potential is periodic in x and y

$$V(x+a_1, y)=V(x, y+a_2)=V(x, y) \tag{21}$$

with a_1 and a_2 being the unit lattice spacings. The canonical momenta p_i's are related to π_i's by

$$\pi_i=p_i-\frac{e}{c}A_i. \tag{22}$$

A is the vector potential satisfying $\varepsilon_{ij}\partial^i A^j=B$ whose general solution is

$$A_i=-\frac{B}{2}\varepsilon_{ij}x^j+\partial_i\Lambda. \tag{23}$$

The scalar Λ is an arbitrary function that determines the gauge we are working in.

If $\boldsymbol{B}$ were to be zero, we would have the well known Bloch electron and the system would have to be invariant under the lattice translation operators

$$\tau_m=e^{(i/\hbar)\boldsymbol{R}_m\cdot\boldsymbol{p}} \tag{24}$$

that act on an arbitrary function according to $\tau_m f(\boldsymbol{r})=f(\boldsymbol{r}+\boldsymbol{R}_m)$. We have $\boldsymbol{m}=(m_1, m_2)\in\boldsymbol{Z}\times\boldsymbol{Z}$ and $\boldsymbol{R}_m=m_1\boldsymbol{a}_1+m_2\boldsymbol{a}_2$ is an arbitrary Bravais lattice vector. This pure spatial translational symmetry arises as a consequence of the periodicity of the potential and leads to a classification of the solutions (Bloch wavefunctions) by means of the eigenvalues $\boldsymbol{p}$ (called the crystal momentum). When a constant magnetic field is present H is no longer invariant under the action of τ_m. The reason is because A is not constant whereas $\boldsymbol{B}$ is. Nevertheless, even in this case there are the so-called magnetic translation operators

$$T_m=e^{(i/\hbar)\boldsymbol{R}_m\cdot\boldsymbol{\beta}} \tag{25}$$

that leave $\boldsymbol{H}$ invariant. $\boldsymbol{\beta}$ is given by

$$\beta_i=\pi_i-\mu\omega\varepsilon_{ij}x^j \tag{26}$$

and its components satisfy the commutation relations

$$[\beta_k, \beta_l]=-[\pi_k, \pi_l]=i\hbar\mu\omega\varepsilon_{kl}. \tag{27}$$

$\boldsymbol{\beta}$, as a constant of motion, is classically connected with the cyclotron centre. $[T_m, \pi^2/2\mu]=0$ in any gauge, however, $[T_m, H]=0$ is valid only in a gauge that is fixed by a function that is at most a quadratic function of the coordinates because only in such a gauge we have $T_m=$ (a phase) τ_m. We gave the explicit factorization of T_m's in some particular Coulomb gauges elsewhere [10]. In order to understand the physical meaning of magnetic translation operators let us define the following dimensionless operators:

$$l_m=\frac{1}{\hbar}\boldsymbol{R}_m\cdot\boldsymbol{\beta} \tag{28}$$

that satisfy

$$[l_m, l_n]=-2\pi i\frac{\phi_{m,n}}{\phi_0} \tag{29}$$

where

$$\phi_{m,n} = -\phi_{n,m} = (\boldsymbol{R}_m \times \boldsymbol{R}_n) \cdot \beta = \boldsymbol{m} \times \boldsymbol{n}\phi_1. \tag{30}$$

$\boldsymbol{m} \times \boldsymbol{n} = m_1 n_2 - m_2 m_1$. $\phi_{m\times n}$ is the magnetic flux that passes through the cell defined by $\boldsymbol{R}_m$ and $\boldsymbol{R}_n$, and

$$\phi_1 = (\boldsymbol{a}_1 \times \boldsymbol{a}_2) \cdot \boldsymbol{B} \qquad \phi_0 = \frac{ch}{e} \tag{31}$$

are the magnetic flux through the unit cell $\boldsymbol{a}_1 \times \boldsymbol{a}_2$, and the magnetic flux quantum (fluxon), respectively. We can now write the magnetic translation operators as

$$T_m = \mathrm{e}^{il_m}. \tag{32}$$

Since the commutator $[l_m, l_n]$ is a c-number, using the Baker–Campbell–Hausdorff formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$ it is easily verified that

$$T_m T_n = \mathrm{e}^{\pi i (\boldsymbol{m}\times\boldsymbol{n})\alpha} T_{m+n} \tag{33}$$

$$= \mathrm{e}^{2\pi i (\boldsymbol{m}\times\boldsymbol{n})\alpha} T_n T_m \tag{34}$$

where $\alpha = \phi_1/\phi_0$ is the number of fluxons passing through the unit cell. The parameter α plays a very fundamental role in the subsequent discussions. The following important observation concerning the nature of α is due to Bloch [11]: for a square lattice $a_1 = a_2 = a$, the period of motion of an electron in a state with crystal momentum $2\pi\hbar/a$ is $t_{\text{crystal}} = \mu a^2/\hbar$, and the period of the cyclotron motion is $t_{\text{cyclotron}} = 2\pi/\omega$. Thus, the ratio of the two fundamental periods of the problem is $\alpha = t_{\text{crystal}}/t_{\text{cyclotron}}$.

Equation (34) clearly shows that, although the operators τ_m's form an infinite cyclic group, the magnetic translation operators T_m's do not form such a group, but rather a projective ray group [12]. Moreover, it is easy to show that T_m's generate the following infinite-dimensional Lie algebra:

$$[T_m, T_n] = 2i \sin(\pi\alpha\boldsymbol{m}\times\boldsymbol{n}) T_{m+n}. \tag{35}$$

The Jacobi identity is satisfied. An important point that must be emphasized once again is that our investigation is almost independent of the choice of a gauge. The only condition we require is that the quadratic gauge is needed for H to be invariant under the action of T_m's.

The main properties of the algebra of magnetic translation operators can be stated as follows. T_m's are unitary, i.e. $T_m^\dagger = T_{-m}$, and satisfy the symmetric product

$$[T_m, T_n]_+ = 2\cos(\pi\alpha\boldsymbol{m}\times\boldsymbol{n}) T_{m+n} \tag{36}$$

where $[\ ,\ ]_+$ denotes the anti-commutator. Furthermore our algebra admits two subalgebras generated by

$$T(a_1) = \{ T_{a_1}^{(m_1)} = \mathrm{e}^{(i/\hbar)m_1 a_1 \beta_1} | m_1 \in \boldsymbol{Z} \} \tag{37}$$

$$T(a_2) = \{ T_{a_2}^{(m_2)} = \mathrm{e}^{(i/\hbar)m_2 a_2 \beta_2} | m_2 \in \boldsymbol{Z} \} \tag{38}$$

respectively. These are infinite Abelian groups by themselves, but they do not commute with each other:

$$T_{a_1}^{(m_1)} T_{a_2}^{(m_2)} = \omega^{m_1 m_2} T_{a_2}^{(m_2)} T_{a_1}^{(m_1)}. \tag{39}$$

where $\omega = e^{2\pi i\alpha}$. Therefore

$$[T_{a_1}^{(m_1)}, T_{a_2}^{(m_2)}] = 2i\sin(\pi m_1 m_2 \alpha) T_m. \tag{40}$$

Consequently every T_m admits the following factorization

$$T_m = \omega^{-m_1 m_2/2} T_{a_1}^{(m_1)} T_{a_2}^{(m_2)}. \tag{41}$$

In the limit as $\alpha \to 0$, i.e. the magnetic field $B \to 0$, the algebra of pure spatial translations is recovered: $\lim_{\alpha\to 0}[T_m, T_n] = [\tau_m, \tau_n]$. If the T_m's are rescaled as

$$T'_m = -i(2\pi\alpha)^{-1} T_m \tag{42}$$

then

$$[T'_m, T'_n] = \frac{1}{\pi\alpha}\sin(\pi\alpha \boldsymbol{m}\times\boldsymbol{n}) T'_{m+n} \tag{43}$$

and the algebra $Diff_A T^2$ is recovered in the limit as $\alpha \to 0$:

$$\lim_{\alpha\to 0}[T'_m, T'_n] = \boldsymbol{m}\times\boldsymbol{n} T'_{m+n}. \tag{44}$$

The above properties of the algebra of magnetic translation operators indicate that certain restrictions on α may lead to physically interesting finite subalgebras. From the physical point of view such restrictions on α will result from appropriate boundary conditions on the substrate lattice structure. Boundary conditions that reduce the infinite lattice to a finite one, say of the size $N_1 N_2\, \boldsymbol{a}_1 \times \boldsymbol{a}_2$, so as not to destroy the algebraic structure will serve the purpose. We take a natural generalization of the Born–von Karman boundary conditions that apply to the zero-field case [8]. The Born–von Karman boundary conditions require the Bloch eigenfunctions to go into themselves under pure spatial translations corresponding to the full finite lattice: $\tau_{N_1}|\psi\rangle = \tau_{N_2}|\psi\rangle = |\psi\rangle$. In this case, if one particular eigenfunction goes into itself, then all the other eigenfunctions obtained from this one by arbitrary lattice translations also go into themselves. This is an obvious consequence of the Abelian nature of the generators T_m's. On the other hand, this property in general is no longer valid for T_m's. To see it, let us suppose that $T_{N_1}|\psi\rangle = T_{N_2}|\psi\rangle = |\psi\rangle$. Then, since $[T_m, H] = 0$, the wavevector $|\psi_m\rangle = T_m|\psi\rangle$ is also an eigenvector of the Hamiltonian. However.

$$T_{N_1}|\psi_m\rangle = e^{2\pi i N_1 m_2 \alpha}|\psi_m\rangle, \quad T_{N_2}|\psi_m\rangle = e^{-2\pi i N_2 m_1 \alpha}|\psi_m\rangle \tag{45}$$

so that in general $T_{N_i}|\psi_m\rangle \neq |\psi_m\rangle$. From this it follows that the so-called magnetic boundary conditions can be satisfied by all eigenfunctions if and only if

$$N_1\alpha = (\text{integer}) \qquad N_2\alpha = (\text{integer}) \qquad \frac{N_1 N_2 \alpha}{2} = (\text{integer}). \tag{46}$$

Thus for the irrational values of α the magnetic boundary conditions cannot be imposed and consequently our algebra has no finite subalgebra. The case with the integer values of α is of no physical interest, since in this case we have nothing other than the Abelian algebra of τ_m's. Furthermore the integer values of α are practically very difficult to obtain because this restriction is in fact a restriction on the magnetic field strength and for a normal sized unit cell we would have

$$\beta \simeq (\text{integer})\frac{4.136\times 10^9}{a_1 a_2(\text{in units } \text{Å}^2)} \quad \text{Gauss}. \tag{47}$$

So the only remaining case is

$$\alpha=\frac{\phi_1}{\phi_0}=\begin{cases}\dfrac{l}{N} & N\text{ even}\\[2mm] 2\dfrac{l}{N} & N\text{ odd}\end{cases} \tag{48}$$

where l and N are relatively prime integers. Then the conditions (46) can be satisfied simultaneously provided N_1 and N_2 are two integers such that N is their greatest common divisor. That is to say, magnetic boundary conditions cannot be imposed for relatively prime N_1 and N_2. Suppose they are not relatively prime. Then two subcases must be distinguished: (A) when $N_1=N_2=N$ and (B) when $N_1=M_1N$, $N_2=M_2N$ with M_1 and M_2 being relatively prime.

Case (A). $N_1=N_2=N$
In this case we have modulo-N structure in the argument of the structure constants and

$$T_{a_1}^N=T_{a_2}^N=1 \qquad T_{a_1}T_{a_2}=\omega T_{a_2}T_{a_1} \tag{49}$$

where $\omega=e^{2\pi il/N}$. Comparing these with the equations (13) we can identify T_{a_1} with the matrix h and T_{a_2} with the matrix g. Thus the magnetic translation operators subjected to magnetic boundary conditions generate an $SU(N)$ algebra when N is odd, and $SU(N/2)$ algebra when N is even.

Case (B). $N_1=M_1N$, $N_2=M_2N$ (M_1 and M_2 relatively prime integers)
In this case without loss of generality we can set $M_2=1$, since the structure constants in (35) do not chance with the replacement $(m_1, tm_2)\to(tm_1, m_2)$. Thus we may identify $T_{(m_1+k_1M_1N,\, m_2+k_2N)}=T_{(m_1,m_2)}$ so that M_1N generators split into mutually commuting M_1 $SU(N)$ algebras (i.e. M_1 factors $SU(N)\times\ldots\times SU(N)$). We may label $SU(N)$ algebra by an index $q=0, 1, \ldots, M_1-1$:

$$SU(N)^q=\left\{T^{(q)}_{(m_1,m_2)}=\frac{1}{M_1}\sum_{p=0}^{M_1-1}e^{2\pi ipq/M_1}T_{(m_1+pN,\, m_2)}\middle| m_1, m_2\in \boldsymbol{Z}\times\boldsymbol{Z}\right\}. \tag{50}$$

It can be easily shown that

$$[T^q_{\boldsymbol{m}}, T^{q'}_{\boldsymbol{n}}]=2i\sin\left(\frac{\pi l}{N}\boldsymbol{m}\times\boldsymbol{n}\right)T^q_{\boldsymbol{m}+\boldsymbol{n}}\delta^{qq'}. \tag{51}$$

The above considerations can be easily extended to the case of many Bloch electrons [13]. Furthermore, in this case we may let the Bloch electrons interact via a potential that depends only on the relative distances between the particles. We take the Hamiltonian

$$H=\sum_{\gamma=1}^{N_e}\left(\frac{\pi^{(\gamma)2}}{2\mu}+V_0(r^{(\gamma)})\right)+\sum_{\gamma<\eta}V(r^{(\eta)}-r^{(\gamma)}) \tag{52}$$

where N_e is the number of electrons and the superscripts label the particles. The substrate potential V_0 is periodic in x and y:

$$V(x^{(\gamma)}+a_1, y^{(\gamma)})=V(x^{(\gamma)}, y^{(\gamma)}+a_2)=V(x^{(\gamma)}, y^{(\gamma)}). \tag{53}$$

Similar to before, we obtain the following algebra of 1-particle operators:

$$[\pi_k^{(\gamma)}, \pi_l^{(\eta)}] = \mathrm{i}\hbar\mu\omega\,\varepsilon_{kl}\delta^{\gamma\eta} \tag{54}$$

$$[\beta_k^{(\gamma)}, \beta_l^{(\eta)}] = -\mathrm{i}\hbar\mu\omega\,\varepsilon_{kl}\delta^{\gamma\eta} \tag{55}$$

$$[\pi_k^{(\gamma)}, \beta_l^{(\eta)}] = 0 \tag{56}$$

where

$$\pi_k^{(\gamma)} = -\mathrm{i}\hbar\partial_k^{(\gamma)} - \frac{e}{c}A_k^{(\gamma)} \tag{57}$$

$$\beta_k^{(\gamma)} = \pi_k^{(\gamma)} - \mu\omega\,\varepsilon_{kl}x^{l(\gamma)} \tag{58}$$

and for any γ

$$\varepsilon^{ij}\partial_i^{(\gamma)}A_j^{(\gamma)} = B. \tag{59}$$

The 1-particle magnetic translation operators $T_m^{(\gamma)}$ generate the algebra

$$[T_m^{(\gamma)}, T_n^{(\eta)}] = 2\mathrm{i}\sin(\pi\alpha\boldsymbol{m}\times\boldsymbol{m})T_{m+n}^{(\gamma)}\delta^{\gamma\eta}. \tag{60}$$

Then the magnetic translation operators for N_e electrons is defined by

$$\tilde{T}_m = \prod_{\gamma=1}^{N_e} T_m^{(\gamma)} = \exp\left(\frac{\mathrm{i}\boldsymbol{R}_m}{\hbar}\cdot\sum_{\gamma=1}^{N_e}\beta(\gamma)\right). \tag{61}$$

Making use of (60) we can show that

$$[\tilde{T}_m, \tilde{T}_n] = 2\mathrm{i}\sin(\pi\alpha'\boldsymbol{m}\times\boldsymbol{n})\tilde{T}_{m+n} \tag{62}$$

where $\alpha' = N_e\alpha$. Then $\tilde{T}_m$'s are symmetry operators of the many-electron Hamiltonian H given by (52) in any gauge determined by at most a quadratic function of coordinates.

4. A realization of the supersymmetric sine algebra

We next consider a Bloch electron with spin-$\frac{1}{2}$ in a constant uniform magnetic field. The Hamiltonian in this case is

$$H = \frac{(\sigma\cdot\pi)^2}{2\mu} + V(x, y) = \frac{\delta_{ij}\pi^i\pi^j}{2\mu} - \frac{1}{2}\hbar\omega\sigma_z + V(x, y) \tag{63}$$

where σ_i are the Pauli spin matrices and $\sigma = (\sigma_x, \sigma_y)$. Without the periodic potential $V(x, y)$, the problem is supersymmetric [14] and the eigenfunctions of the Hamiltonian are labelled, in addition to the usual bosonic quantum numbers $(n, m)_B$, by a fermionic quantum number $n_F = 0, 1$, i.e. $|(n, m)_B, n_F\rangle$. This new quantum number introduces a two-fold degeneracy of each excited state in addition to the infinite degeneracy of the well known Landau states. In order to obtain the symmetry generators of H given by (63) all we have to do is to multiply the magnetic translation operators T_m's by a 2×2 matrix that commutes with σ_z [10]. The most general form of such a matrix is $c_1 a + c_2 b$, $c_1, c_2 \in \mathbf{C}$ where

$$a = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \qquad b = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{64}$$

such that $a^2=a$, $b^2=b$, $ab=ba=0$. Then we define the non-unitary operators

$$T_m^{(A)}=aT_m,\ T_m^{(B)}=bT_m \tag{65}$$

that satisfy the commutation relations

$$[T_m^{(i)}, T_n^{(j)}]=2\mathrm{i}\sin(\pi\alpha \boldsymbol{m}\times\boldsymbol{n})T_{m+n}^{(i)}\delta^{ij} \qquad i,j=A,B. \tag{66}$$

One can construct unitary operators by taking the linear combinations

$$T_m^{(\pm)}=T_m^{(A)}\pm T_{m^*}^{(B)}. \tag{67}$$

where the * map of an integral 2-vector $\boldsymbol{m}$ is defined as follows ($\varepsilon=\pm 1$):

$$\boldsymbol{m}^{**}=\boldsymbol{m} \qquad (\boldsymbol{m}+\boldsymbol{n})^*=\boldsymbol{m}^*+\boldsymbol{n}^* \qquad \boldsymbol{m}^*\times\boldsymbol{n}^*=\varepsilon\boldsymbol{m}\times\boldsymbol{n}. \tag{68}$$

Some examples of * maps of $\boldsymbol{m}$ are given for $\varepsilon=1$

$$\boldsymbol{m}^*=\boldsymbol{m} \qquad \boldsymbol{m}^*=-\boldsymbol{m} \tag{69}$$

and for $\varepsilon=-1$

$$\boldsymbol{m}^*=(m_2, m_1) \qquad \boldsymbol{m}^*=(-m_1, m_2) \qquad \boldsymbol{m}^*=(m_1, -m_2). \tag{70}$$

It is easily verified that provided $\boldsymbol{m}^*=\boldsymbol{m}$ we have

$$T_m^{(+)}=\mathrm{e}^{\mathrm{i}l_m 1}=\begin{pmatrix} L_m & 0 \\ 0 & L_m \end{pmatrix} \tag{71}$$

while for $\boldsymbol{m}^*=-\boldsymbol{m}$ we have

$$T_m^{(+)}=\mathrm{e}^{\mathrm{i}l_m\sigma_2}=\begin{pmatrix} L_m & 0 \\ 0 & L_{-m} \end{pmatrix}. \tag{72}$$

Then the choice $\varepsilon=1$ leads to the following algebra commutators:

$$[T_m^{(+)}, T_n^{(+)}]=[T_m^{(-)}, T_n^{(-)}]=2\mathrm{i}\sin(\pi\alpha\boldsymbol{m}\times\boldsymbol{n})T_{m+n}^{(+)} \tag{73}$$

$$[T_m^{(+)}, T_n^{(-)}]=2\mathrm{i}\sin(\pi\alpha\boldsymbol{m}\times\boldsymbol{n})T_{m+n}^{(-)} \tag{74}$$

together with the symmetric bracket relations

$$[T_m^{(+)}, T_n^{(+)}]=[T_m^{(-)}, T_n^{(-)}]_+=2\cos(\pi\alpha\boldsymbol{m}\times\boldsymbol{n})T_{m+n}^{(+)} \tag{75}$$

$$[T_m^{(-)}, T_n^{(+)}]_+=2\cos(\pi\alpha\boldsymbol{m}\times\boldsymbol{n})T_{m+n}^{(-)}. \tag{76}$$

All the extended operators above commute with the Hamiltonian $\boldsymbol{H}$ given by (63). Let us define a set of rescaled generators:

$$K_m=(2\mathrm{i}\pi\alpha)^{-1}T_m^{(+)} \tag{77}$$

$$F_m=(4\mathrm{i}\pi\alpha)^{-1/2}T_m^{(-)}. \tag{78}$$

These span the superalgebra

$$[K_m, K_n]=\frac{1}{\pi\alpha}\sin(\pi\alpha\boldsymbol{m}\times\boldsymbol{n})K_{m+n} \tag{79}$$

$$[K_m, F_n]=\frac{1}{\pi\alpha}\sin(\pi\alpha\boldsymbol{m}\times\boldsymbol{n})F_{m+n} \tag{80}$$

$$[F_m, F_n]_+ = \cos(\pi\alpha m \times n) K_{m+n}. \tag{81}$$

Thus we have obtained the supersymmetric extension of the sine algebra and its explicit physical realization in terms of magnetic translation operators. The limit $\alpha \to 0$ yields as one would have expected the supersymmetric generalization of $Diff_A T^2$:

$$[L_m, L_n] = m \times n L_{m+n} \tag{82}$$

$$[L_m, G_n] = m \times n G_{m+n} \tag{83}$$

$$[G_m, G_n]_+ = L_{m+n}. \tag{84}$$

5. Conclusions

We have demonstrated that the motion of a Bloch electron in a constant uniform magnetic field that is commensurate with the substrate lattice provides an example from condensed matter physics where the sine algebra (1) makes its appearance. There are some aspects of this problem that we think are worthy of remark. First, it should be emphasized that the concept of a Lie algebra deformation has nothing intrinsic to do with quantization of a classical system. In fact we have shown here that it is α, the number of fluxons that pass through a unit cell, that plays the role of the deformation parameter. The derivation of the sine algebra can be carried out with equal ease either at the classical level using the Poisson brackets or at the quantum level using canonical commutation relations. The magnetic translation operators T_m's that span the algebra (1) would have been obtained exactly as above had we worked in terms of Poisson brackets among the classical phase space variables.

In the last section we gave the explicit physical realization of the supersymmetric sine algebra in terms of magnetic translation operators. This construction is new and it might be relevant to the study of anyons in a magnetic field [15], and hence add to the current theoretical understanding of the quantized Hall effect [16].

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